Chaos Control of the Hénon map and an Impact Oscillator by the Ott-Grebogi-Yorke Method

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Abstract

In control systems, the prescence of chaos is usually regarded as a nuisance and is designed out of the system if possible. However, the Ott-Grebogi-Yorke method is a method of feedback control which takes advantage of chaos in a system to stabilize an unstable periodic orbit. In this report, the OGY method is applied to two systems, the Hénon map and a map describing the motion of an impact oscillator, to control a period-1 orbit.

1 Introduction

1.1 Basic Control Systems

Control systems are designed to improve performance or accuracy of a process, and are used across a wide range of disciplines. They work by sustaining an operating environment within certain limits by regulating a particular variable as accurately as possible. Some examples of control systems are control of temperature, control of a simple electrical circuit, with an input and output voltage, or control of viscosity in fuel oil. [3, 4] Control systems are divided into two categories, feedback and nonfeedback control. In nonfeedback control, the controlled variable is simply output after it has been acted upon, while in feedback control, the size of the controlled variable is measured and fed back into the system.

1.2 Chaos Control Systems

In engineering systems, chaos occurs frequently and is usually designed out if possible. In chaos control, the presence of chaos is taken advantage of to help bring the system to stability. In some situations, chaos may actually be designed into the system and then exploited to reach the desired control [2]. Like the general control system described above, chaos control falls into two camps, feedback and nonfeedback control methods.

Feedback control is often used as an alternative to the common method of completely altering the dynamics of the system in order to eliminate the presence of chaos. Although these alterations may still ultimately result in the desired behavior, the alteration might be extremely costly, or simply not available for some reason. In feedback control, a particular trajectory in the phase space is targeted, and a feedback process is applied to the trajectory to maintain control. The important features of a feedback control method is that the original system remains unchanged while unstable periodic orbits on the the chaotic attractors are stabilized.

In nonfeedback control methods, the main idea is to change the behavior of the system from that of a chaotic attractor to a nearby periodic orbit. These changes may be done in several ways, commonly by making some small permanent change to a control parameter. Chaos can also be controlled by a system design in which a chaotic system may be coupled to a simpler asymptotically stable system [2]. In general, the nonfeedback methods require more knowledge of the equations of motion of the system, but unlike the feedback methods, have the advantage of not needing to follow a particular trajectory.

2 Feedback Control: Ott-Grebogi-Yorke Method

2.1 Overview

The OGY method was the pioneering method of feedback control. Since then, many variations of the OGY method have been used successfully to control various systems. The main idea of the method is to seek improved performance of a system by making small time-dependent perturbations in order to obtain an attracting time-periodic motion [6]. It is a very general method in which a discrete map can be used and equations of motion are not required.

This method takes advantage of three characteristics of a chaotic attractor in order to achieve control [5]. Because a chaotic attractor contains an infinite number of unstable periodic orbits, the method provides flexibility in choosing which of the unstable orbits to stabilize. A particular unstable orbit may be chosen because it gives the optimal performance out of a subset of unstable orbits, or because of the particular goal behavior that is required for a given set of parameter values.

The second characteristic of a chaotic attractor that is used is the instability property, that is, the sensitive dependence on initial conditions. The method takes advantage of this by applying very small perturbations to adjust the system dynamics. The only requirement is that the perturbations are small enough so as not to introduce new trajectories altogether [7].

The final aspect of chaotic systems that is exploited to achieve control is the existence of dense orbits. In the OGY method, control is applied when a particular trajectory enters a band around the unstable orbit to be stabilized. Since dense orbits exist in the chaotic attractor, we are guaranteed that at some point in time the trajectory will enter the band and control can be applied. Following is a simple overview of the OGY method [6]:

1. Determine some of the unstable low-period periodic orbits which are embedded in the chaotic attractor

2. Examine these orbits, and choose one which yields improved system performance

3. Adjust small time-dependent parameter perturbations in order to stabilize the chosen unstable periodic orbit

There are two main disadvantages of the method. The method relies on the fact that the trajectory being monitored will eventually come within a particular band around the unstable periodic orbit, at which point control can be applied. However, the time that the trajectory takes to enter that band may be extremely long. Because of this, techniques have been developed which direct a trajectory to a neighborhood of the unstable attractor in order to decrease the length of time before control can be applied [2]. The other disadvantage to the method is that small amounts of noise can cause the trajectory to be "kicked" out of the region where the control is on. However, if the mean time between kick-outs is small enough, then control can still be obtained [7].

2.2 Procedure

To begin describing the OGY method in detail, we assume we have a discrete time system given by

$$\mathbf{Z}_{i+1} = F(\mathbf{Z}_i, p), \quad \mathbf{Z}_i \in \Re^n, \ p \in \Re$$

where we require $|p - \bar{p}| < \delta$ for some nominal value $\bar{p}$, where the system has a chaotic attractor at $p = \bar{p}$. If the system dynamics are described by continuous equations, then a discrete system can be found by taking a Poincaré section. The goal is to vary the parameter p so that for almost all initial conditions in the basin of the chaotic attractor, the system converges onto the desired periodic orbit. The control strategy is as follows [6, 7]:

1. Find a stabilizing local feedback control law defined on a neighborhood of the desired periodic orbit. We will consider the first order approximation of the system at the chosen unstable periodic orbit, and assume that the approximation is stabilizable.

2. Due to chaotic dynamics, we are ensured that the trajectory will eventually enter the neighborhood.

3. Once the trajectory is inside the neighborhood, we apply stabilizing feedback control to steer the trajectory towards the desired orbit.

Define $\mathbf{Z}_*(\bar{p})$ to be the unstable fixed point in the attractor. For p close to $\bar{p}$ and in the neighborhood of the fixed point $\mathbf{Z}_*(\bar{p})$, we can approximate the map by

$$\mathbf{Z}_{i+1} - \mathbf{Z}_*(\bar{p}) = \mathbf{A}[\mathbf{Z}_i - \mathbf{Z}_*(\bar{p})] + \mathbf{B}(p - \bar{p})$$

$$\begin{aligned} \text{where } \mathbf{A} &= \mathbf{D}_Z F(\mathbf{Z}, p) \\ \text{and } \mathbf{B} &= \partial_p F(\mathbf{Z}, p). \end{aligned}$$

The matrix $\mathbf{A}$ can be decomposed as follows:

$$\mathbf{A} = \lambda_u \mathbf{e}_u \mathbf{f}_u^T + \lambda_s \mathbf{e}_s \mathbf{f}_s^T$$

where $\{\mathbf{e}_u, \mathbf{e}_s\}$ are the eigenvectors corresponding to the unstable and stable eigenvalues (λ_u, λ_s), and $\{\mathbf{f}_u, \mathbf{f}_s\}$ are the contravariant eigenvectors such that

$$\mathbf{f}_s^T \mathbf{e}_s = \mathbf{f}_u^T \mathbf{e}_u = 1, \ \mathbf{f}_s^T \mathbf{e}_u = \mathbf{f}_u^T \mathbf{e}_s = 0.$$

In order to control the system to the desired fixed point, the idea is to adjust the parameter p so that the point $\mathbf{Z}_{i+1}$ falls on the stable manifold of $\mathbf{Z}_*(\bar{p})$. To do this, we need to have

$$\mathbf{f}_u^T (\mathbf{Z}_{i+1} - \mathbf{Z}_*(\bar{p})) = 0.$$

Using the linearization around the fixed point and the decomposition of $\mathbf{A}$, we have that

$$p - \bar{p} = -\mathbf{K}^T [\mathbf{Z}_{i+1} - \mathbf{Z}_*(\bar{p})]$$

where $\mathbf{K}$ is defined by

$$\mathbf{K} \equiv \frac{\lambda_u}{\mathbf{f}_u^T B}\mathbf{f}_u$$

and we assume $\mathbf{f}_u^T \mathbf{B} \neq 0$.

Since we require $|p - \bar{p}| < \delta$ for some chosen δ, and $p - \bar{p} = -\mathbf{K}^T[\mathbf{Z}_i - \mathbf{Z}_*(\bar{p})]$, we have $|\mathbf{K}^T[\mathbf{Z}_i - \mathbf{Z}_*(\bar{p})]| < \delta$. We would like to activate control only for values of $\mathbf{Z}_i$ which are inside the width of the slab $2\delta/|\mathbf{K}^T|$, and leave $p = \bar{p}$ when $\mathbf{Z}_i$ is outside the slab. We then have that control is determined by

$$p - \bar{p} = \begin{cases} -\mathbf{K}^T[\mathbf{Z}_i - \mathbf{Z}_*(\bar{p})] & \text{if } |\mathbf{K}^T[\mathbf{Z}_i - \mathbf{Z}_*(\bar{p})]| < \delta \\ 0 & \text{otherwise} \end{cases}$$

The general idea of the method is to start with some initial condition resulting in a chaotic orbit. Since we are assuming that the range of p will allow the system to converge to the desired periodic orbit, we let the orbit continue until $\mathbf{Z}_i$ falls within the slab $2\delta/|\mathbf{K}^T|$. At that point, control can be applied. Due to the nonlinearities of $F(\mathbf{Z}, p)$, it is possible that $\mathbf{Z}_i$ may enter the slab only to continue wandering chaotically outside of it. However the trajectory will eventually re-enter the slab and be able to be controlled.

3 Examples

3.1 Henon Map

In this example, the OGY method will be applied to the Hénon map, which is given by:

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} a - x_n^2 + by_n \\ x_n \end{pmatrix}$$

When $b = 0.3$, this map has a chaotic attractor for parameter $\bar{a} = 1.4$. The fixed point for this map is given by

$$x_* = y_* = -c + \sqrt{c^2 + \bar{a}}$$

where $c = \frac{1}{2}(1 - b)$. The steps in controlling this system are as follows [7]:

1. Identify the periodic orbit to be stabilized
2. Calculate $\mathbf{Z}_*(\bar{a})$:

$$\mathbf{Z}_*(\bar{a}) = \begin{pmatrix} x_* \\ y_* \end{pmatrix} = \begin{pmatrix} -c + \sqrt{c^2 + \bar{a}} \\ -c + \sqrt{c^2 + \bar{a}} \end{pmatrix} = \begin{pmatrix} .8839 \\ .8839 \end{pmatrix}$$

3. Calculate matrices $\mathbf{A}$ and $\mathbf{B}$:

$$\mathbf{A} = \begin{pmatrix} -2x_* & b \\ 1 & 0 \end{pmatrix}, \ \mathbf{B} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

4. Calculate λ_u, λ_s:

$$\begin{aligned} \lambda_{u,s} &= -x_* \pm \sqrt{x_*^2 + b} \\ \lambda_u &= -1.9327 \\ \lambda_s &= .1559 \end{aligned}$$

5. Calculate $\{\mathbf{e}_u, \mathbf{e}_s\}$ and $\{\mathbf{f}_u, \mathbf{f}_s\}$:

$$\begin{pmatrix} \mathbf{e}_u & \mathbf{e}_s \end{pmatrix} = \begin{pmatrix} -0.8873 & -0.1541 \\ 0.4612 & -0.9881 \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{f}_u \\ \mathbf{f}_s \end{pmatrix} = \begin{pmatrix} -1.0425 & 0.1626 \\ -0.4867 & -0.9362 \end{pmatrix}$$

6. Calculate $\mathbf{K}$:

$$\mathbf{K} = \begin{bmatrix} \lambda_u & -\lambda_u \lambda_s \end{bmatrix} = \begin{bmatrix} -1.9327 & .3 \end{bmatrix}$$

7. Select δ

To apply control to the system, the algorithm is applied as follows:

1. Let $j = 0$. Pick an initial point $\mathbf{Z}_j$.
2. Calculate $-\mathbf{K}^T[\mathbf{Z}_j - \mathbf{Z}_*(\bar{a})]$ to find a_j.
3. Calculate $\mathbf{Z}_{j+1}$. Update j.
4. Continue until $\mathbf{Z}_j$ remains close to $\mathbf{Z}_*(\bar{a})$ for all j greater than some value $j_{\max}$.

3.2 Impact Oscillator

This second example demonstrates the control of chaos in an impact oscillator, which is a spring-mass sytem which impacts with a sinusoidally vibrating table [1]. The difference equations for this system are defined by

$$\begin{pmatrix} \Omega_{i+1} \\ \Phi_{i+1} \end{pmatrix} = \begin{pmatrix} \Omega_i + \rho \tan^{-1}(\Phi_i) \\ \alpha\Phi_i - \rho\gamma\cos(\Omega_j + \rho\tan^{-1}(\Phi_i)) \end{pmatrix}$$

where the control parameter is given by ρ. The mapping is 2π periodic in Ω, so that the phase space is $S^1 \times \Re$. The oscillator is chaotic for parameter value $\bar{\rho} = 7.1$ with $\gamma = .225$ and $\alpha = 0.8$. We have the following data for the control:

1. $\mathbf{Z}_*(\bar{\rho}) = \begin{pmatrix} \Omega_* \\ \Phi_* \end{pmatrix} = \begin{pmatrix} 1.72438 \\ 1.22195 \end{pmatrix}$
2. $\mathbf{A} = \begin{pmatrix} 1 & 2.8478 \\ 1.5787 & 5.2958 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 0.8850 \\ 1.4315 \end{pmatrix}$
3. $\lambda_u = 6.1661, \lambda_s = .1297$
4. $\begin{pmatrix} \mathbf{e}_u & \mathbf{e}_s \end{pmatrix} = \begin{pmatrix} -0.4933 & -0.9563 \\ -0.8949 & -0.2922 \end{pmatrix}$

   $\begin{pmatrix} \mathbf{f}_u \\ \mathbf{f}_s \end{pmatrix} = \begin{pmatrix} -0.2922 & -0.9563 \\ -0.8949 & 0.4933 \end{pmatrix}$
5. $\mathbf{K} = \begin{pmatrix} 1.1071 & 3.6230 \end{pmatrix}$
6. $\delta = .01$

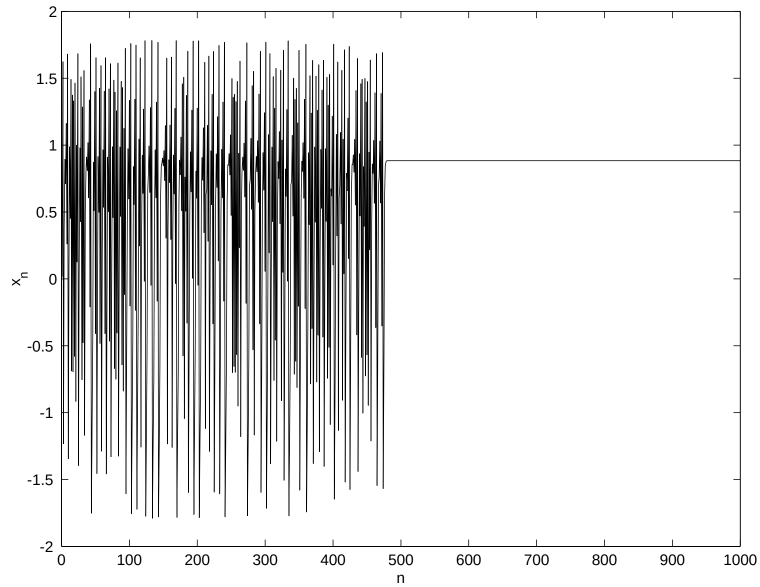

Figure 1: Control of an unstable fixed point in the Hénon map

4 Results and Conclusions

The OGY method was applied to two maps exhibiting chaos, and the unstable period-1 fixed point in both cases was able to be stabilized. The algorithms were coded in Matlab (see Appendix for Hénon map algorithm). For the Hénon map, it can be seen in Figure 1 that control was achieved in just under 500 iterations at $x = 0.8839$ when $\delta = .01$. Along the vertical axis, the value of x_n at each iteration is shown, and along the horizontal axis, n refers to the number of iterations.

The impact oscillator was able to be controlled around the period-1 orbit $(1.72438, 1.22195)$ by 3000 iterations as can be seen in Figure 2. The figure plots the number of iterations against the second coordinate, Φ, of the oscillator. The impact oscillator also has a period-2 orbit given by the points (Ω, Φ) = (3.6724,2.00164), (5.23236,.780014), and it was attempted to control this period-2 orbit by looking at the second return map. Although the trajectory would enter the window around one of the fixed points, the applied control did not cause the next point to remain in the window. The reason for this was not clear. Due to the complexity of the equations in the second return map, it is possible that there was enough numerical round-off error that the orbit was not able to be controlled. It is also a possibility that since the OGY method can take a long time to achieve control, that the algorithm was simply not run for enough iterations. If this is the case, then a targeting procedure can be implemented in which the trajectory is directed to the band around which control is applied, in order to achieve control in a shorter period of time.

From this point, it might be necessary to try a variation of the Ott-Grebogi-Yorke Method or try a targeting procedure in order to control the period-2 orbit in the impact oscillator. If this could be achieved, then the next step would be to see if control could be switched from a period-1 to a period-2 orbit and back again to a period-1 orbit by adjusting the control parameter during the algorithm.

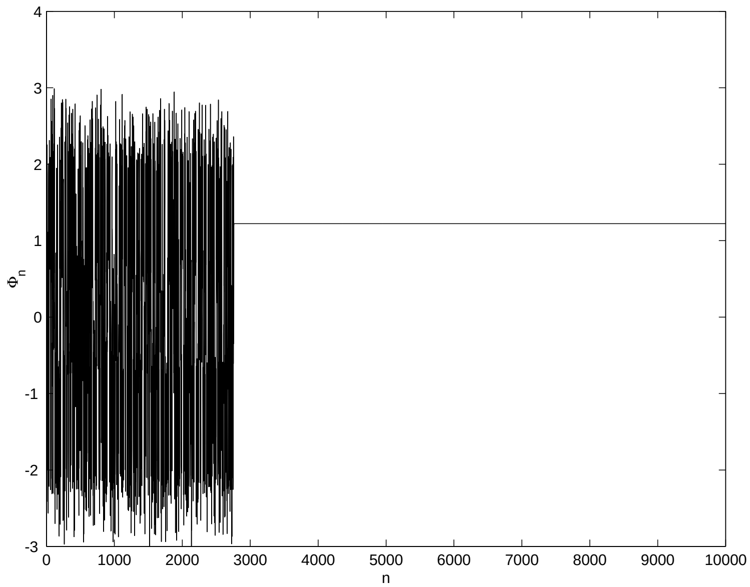

Figure 2: Control of an unstable fixed point in an impact occillator

References

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A Matlab Code

```
function f = henon(iters,delta)
a0 = 1.4;       %value at which Henon map is chaotic
b = .3;
c = .5*(1-b);

z_fp = (-c + sqrt(c^2 + a0))*ones(2,1);  %value of map at fixed point
x_fp = z_fp(1);                          %x value of map at fixed point

lu = -x_fp - sqrt(x_fp^2 + b);           %unstable eigenvalue
ls = -x_fp + sqrt(x_fp^2 + b);           %stable eigenvalue

K = [lu ;-lu*ls]

z(1,:) = rand(1,2);                      %random initial starting point

for j = 1:iters - 1

  if abs(K'*(z(j,:)' - z_fp)) < delta    %if trajectory is in control
                                         %region update parameter a
    ch_a = -K'*(z(j,:)' - z_fp);
  else
    ch_a = 0;                            %otherwise leave as value a = a0
  end

a = a0 + ch_a;

newpoint = a - z(j,1)^2 + b*z(j,2);      %find next point
z(j+1,:) = [newpoint z(j,1)];

end

plot(1:iters,z(:,1))
xlabel('n')
ylabel('x')
```